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CONNECTING ELEMENTARY TEACHERS' MATHEMATICAL KNOWLEDGE TO THEIR INSTRUCTIONAL PRACTICES

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This paper summarizes the results of an investigation of in-service teachers in elementary mathematics education. Of particular interest are the mathematical understandings and instructional practices of 50 teachers with respect to number. Elementary teachers' mathematical knowledge has been shown to be lacking and, therefore, an impediment to their students' understanding of mathematics. In this study, I analyzed the growth of elementary teachers' content and pedagogical content knowledge from three schools engaged in professional development and compared their knowledge to their instructional practices. There was a strong relationship between the gains in content knowledge teachers made over time and their instructional practices.
Keywords: *Content knowledge, pedagogical content knowledge, instructional practices*

THEORETICAL PERSPECTIVES

When serious discussions about curriculum, instruction, and assessment standards for the teaching of mathematics began in the 1980's, mathematics educators also began discussing the possible ramifications of reform on teachers' knowledge of mathematics and their approach to teaching it. To teach to such standards, a teacher needs knowledge of how mathematics is constructed and connected and knowledge of how students might think (informally and formally) about mathematics (Gravemeijer, 2004). Unfortunately, most elementary teachers have not experienced mathematics in this manner (Cohen & Ball, 1990; Knapp & Peterson, 1995) and after two decades of working within the mathematics reform movement, elementary teachers are still entering the teaching field unprepared to teach mathematics in the way envisioned by these standards (Frykholm, 1996; Zeichner, 1993). And due to this lack of exposure to rich mathematical experiences and because it becomes increasingly difficult for teachers to acquire the knowledge, skills, and dispositions to teach in ways aligned with the reform once out in the field (Cuban, 1990; Frykholm, 1996; Richardson, 1990), it becomes critical for elementary teachers to be in situations where they can experience mathematics in

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the aforementioned ways (Grouws & Cebulla, 2000).

TEACHERS' MATHEMATICAL AND PEDAGOGICAL KNOWLEDGE

Many researchers have written about the different types of knowledge teachers should have in order to teach for understanding (Ball, 1989; Fennema & Romberg, 1999; Grossman, Wilson, & Shulman, 1989; McDiarmid, Ball, & Anderson, 1989; Mewborn, 2000; Putnam, Heaton, Prawat, & Remillard, 1992). Shulman (1986) describes two types of knowledge important for teachers to have. First, knowledge of the subject matter is needed in order to navigate the substantive structures organizing the concepts within mathematics and the syntactic structures categorizing the rules that govern it. This describes teachers' procedural and conceptual knowledge. The second form of knowledge is pedagogical content knowledge. It is understanding the ways mathematics can be represented as well as understanding the trajectory of students' informal mathematical strategies and how to help them move toward more formal structures. This knowledge enables a teacher to present the mathematics to students in a way that maximizes understanding and minimizes mistakes and misconceptions. When teachers have these types of knowledge structures, the next step is to focus on their instructional practices. In this case, teachers' instruction is focused on teaching for understanding.

TEACHING FOR UNDERSTANDING

Understanding is described as knowing *how* to do something and *why*. Knowing why enables us to use concepts flexibly, extend our knowledge to new situations, and connect it to the world outside of school (Ball, 1989; Hiebert et al., 1997; McDiarmid & Ball, 1989; McDiarmid et al., 1989; NCTM, 2000; Perkins, 1993). However, conceptual understanding is rarely seen in U.S. mathematics classrooms and, therefore, content is taught as isolated rules or procedures to be memorized (Ma, 1999; Mewborn, 2000; Stigler & Hiebert, 1999). Teaching conceptually helps students extend their knowledge to new situations (Grouws & Cebulla, 2000; Hiebert et al., 1997). If mathematical ideas are only seen in isolation it is unlikely students will use their previous knowledge in new situations (Ball, 1989; Borko et al., 1992; Ma, 1999; Newmann & Associates, 1996).

Many studies show that the quality of what and how a teacher teaches is influenced by the teacher's knowledge (Ball, Lubienski, & Mewborn, 2001; Ball & Mosenthal, 1990; Borko et al., 1992; Grossman et al., 1989; Ma, 1999; McDiarmid et al., 1989; Mewborn, 2000; Putnam et al., 1992). If this is so, then teachers with limited knowledge of the mathematics they teach, will unlikely teach for understanding. For instance, Ma (1999) found that teachers who expected their students to only know mathematical procedures tended to have only procedural knowledge themselves.

OBSERVING TEACHING FOR UNDERSTANDING

In order to capture whether teachers are teaching conceptually, I combined Hiebert et al.'s model for teaching for understanding with research on mathematical discourse. Their 1997 framework suggests there are five core features within a classroom that determine whether teaching for understanding is present: (1) learning tasks are seen as opportunities for students to explore mathematics, (2) the role of the teacher is that of a facilitator, (3) the culture of the classroom is one of a community in which students interact with each other about the topic, (4) mathematical tools are used to deepen students' understanding, and (5) the mathematics is accessible and equitable for all students (Stigler & Hiebert, 1999).

They argue that these five features must be present together for understanding to occur. The nature of the tasks students are asked to perform sets the foundation in the classroom; if the task is procedural, it will be hard for the other four features to surface. Even if the task is one that promotes understanding and not just rote memorization, the teacher's role may negate the effectiveness of the task. For example, a teacher may give a task that encourages students to explore the mathematics but dictates what is to be completed in a step-by-step manner. In order to reinforce students' understanding, the culture of the classroom needs to be one that encourages discourse between teacher and student and among students – which becomes the sixth component of teaching for understanding (Brendefur & Frykholm, 2000; Sherin, 2002). Having students *explain* and *justify* their solution strategy to others helps them think more deeply about their ideas. Before students can communicate what they did, they must reflect and analyze in order to explain it to others.

METHODS AND DATA SOURCES

There are two questions framing this study. The first question is, "How does elementary teachers' Pedagogical Knowledge of Mathematics (PCK) change from before participating in a professional development project to the end of the first year implementation?" The second question is, "What is the relationship between these teachers' pedagogical content knowledge and their instructional practices?"

PARTICIPANTS AND SITUATION

To answer these questions, I partnered with local school districts whose administrators agreed to have their entire faculties participate in the study. The participants were from four elementary schools in three districts: one urban, one suburban, and one rural (see Table 1). Each of the schools had a large percentage of students who were considered at-risk or disadvantaged, having between 57% and 88% of their students on free or reduced lunch and having a relatively high number of English-Language-Learners (ELL) and special education students compared to their district averages.

There were 50 teachers in the study whose data were complete, including pretest and posttest scores, and classroom observations. Teachers' whose data were not complete, due to extraneous circumstances, such as absent from the summer institute or left midyear, were removed from analysis. Table 2 below portrays the demographics of the number of teachers from each school with complete data. There were between 3 and 11 teachers at each grade level. The average number of years taught was 13.8 years (standard deviation of 8.9 years), which included a range of 1 to 35 years.

STUDY CONTEXT AND DATA COLLECTION

This investigation focuses on the first year of a three-year study aimed at Developing Mathematical Thinking (DMT) in teachers to improve their instruction. The professional development included a one-week summer institute in August focusing on the elementary level content areas of number and number operations. This 40 hour institute focused on creating a dialogic discourse in the classroom (Brendefur & Frykholm, 2000; Kazemi & Stipek, 2001) and on how to encourage student sharing and articulation of ideas (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Hiebert et al., 1996) During the institute, teachers worked on deriving different strategies for solving the same problem in order to understand the procedural and conceptual underpinnings of the mathematics. In addition, teachers were taught how to interpret and incorporate strategies that ranged from more informal to more advanced strategies. Teachers were then taught how to change the difficulty of problems based on the student's level of understanding.

Table 1. *School Demographics of DMT Study*

	Belmont S.D.		Cannon S.D.		Elwood S.D.	
School	Trent (K – 6)		Larson (K – 5)		Shadow (K-3)	Carbon (4-6)
Enrollment	369		531		509	409
Faculty	23		21		14	12
Racial/Ethnic Mix	White	89%	White	54%	White	87%
	Latino	7%	Latino	45%	Latino	10%
	Black	1%	Black	1%	Black	1%
	Asian	1%	Asian	0%	Asian	1%
	Native A.	2%	Native A.	0%	Native A.	1%
	Other	0%	Other	0%	Other	0%
ELL/Migrant	2%		16%, 3%		10%	
Languages	Spanish, English		Spanish, English		Spanish, Japanese, English	
Free/reduced lunch	88%		57%		62%	
Special Education	20%		18%		11%	

Table 2. *Demographics of Teachers Participating in the DMT Study*

Variables		Frequency	Percent
Teachers (*n*=50)	Carbon	8	16
	Shadow	12	24
	Larsen	15	30
	Trent	15	30
Grade Level (*n*=50)	Kindergarten	3	6
	First	11	22
	Second	6	12
	Third	6	12
	Fourth	10	20
	Fifth	8	16
	Sixth	4	8
	Specialists	2	4
Variables		Mean	SD
Years Taught (n=50)		13.8	8.93

Ongoing professional development was provided to each of the schools throughout the year. Each school district was assigned one staff member to visit each week. For one school district, this meant that the staff member had to alternate each week between the two schools – Carbon and Shadow. During these weekly visits, the staff member would conduct demonstration lessons, observe the classroom episodes while teachers implemented their lessons, and about every five weeks hold a lesson study meeting for two hours with grade level teachers.

Content Knowledge Inventory

To measure growth in teachers' knowledge of number, a nineteen item knowledge inventory (see Appendix A) was constructed based predominantly on items found in the literature (Ball, 1989; Ball et al., 2001; Ball & Wilson, 1990; Carpenter, Franke, & Levi, 2003; Empson & Junk, 2004; Kennedy, Ball, & McDiarmid, 1993; Lamon, 1999; Ma, 1999; Schifter, 1998, 2001; Sowder, Philipp, Armstrong, & Schappelle, 1998). The items held face validity and were used to gather information on teachers' knowledge of addition, subtraction, multiplication, division and rational number. For each item taken from the literature, the results were compared to the results in the literature. In addition to determining teachers' knowledge of number, the items were also used to determine their pedagogical knowledge. The inventory was administered to participating teachers before any professional development began in early August and, then, again in April. This duration of time was used to determine whether any gains in PCK were retained.

Observation Instrument

During the school year, classroom observations were conducted in each of the classrooms. To show a more accurate portrait of instructional practices, up to three classroom observations were conducted in each of the teachers' classrooms using an observation protocol described above and provided in Appendix B.

Three project staff observed each teacher three times throughout the year – fall, midyear, and spring. Each teacher's practice was rated on each of the six observation attributes. Staff would co-observe approximately every 10th observation. During this observation, raters would rate the lesson independently and then check for reliability, which was (r = 0.92). Any differences would be discussed during bi-monthly staff meetings to ensure future reliability. In addition, qualitative data (lesson transcripts) obtained from the formal observations were used during the staff meetings to provide detailed discussions of what occurred in the classroom episodes.

Analysis & Results

The knowledge tasks were scored using a four-point rubric. Each attribute contained specific elements based on the following general framework: 0 – no response, 1 – incorrect and/or major conceptual flaw, 2 – correct with a minor procedural or conceptual flaw, and 3 – correct with complete conceptual understanding.

Teachers were given a total score for their instruction based on the aggregate scores from six classroom features. If a teacher was observed on more than one occasion, a mean score was calculated. Descriptive statistics were used to examine knowledge gains and Pearson correlations calculated to identify relationships among teacher information, content knowledge and teaching practices.

Changes in Teachers' Content Knowledge

There were over 50 teachers who attended the first summer institute and completed the inventory before the summer institute began. Teachers' mathematical knowledge, based on the 19-item inventory, was very weak, albeit similar to the results found in past research. On average, on any given problem only 39% of the teachers could answer the problem correctly. On 13 of the 19 tasks, over 50% of the teachers either had no response or responded incorrectly, many times with a major misconception.

In March, seven months after the initial institute, the number knowledge inventory was administered to teachers again and analyzed to determine what effect professional development had on the teachers' knowledge. Overall, teachers performed higher on 18 of the 19 items. The paired-samples t-test $t(49) = 9.92$, $p<.001$, indicates that for the 50 teachers, the mean score on the posttest ($M = 39.16$, SD = 7.81) was significantly greater than the mean score on the pretest ($M = 29.74$, SD = 7.82). This gain is quite sizeable with an effect size of 1.21 over the seven month time frame.

Table 4. *Item Analysis for the Results of Each Item on the Tests (n=50)*

Problems	Pre-test	SD	%	Post-test	SD	%	t
16. Addition 65+28	2.24	.744	28	2.88	.328	100	5.56*
3a. Subtraction 36-19	1.58	.906	46	2.54	.503	100	6.55*
3b. Subtraction Justification	1.10	.909	26	1.76	.960	64	3.53*
9. Subtraction Regrouping	1.42	.609	44	2.06	.682	80	4.95
11. Subtraction of Fractions	1.62	.967	60	1.68	.935	60	0.31
1. Multiplication 18x25	2.10	.707	84	2.48	.646	92	2.80*
7. Multiplication 123x645	1.86	.808	64	2.12	.849	74	1.57
15. Multiplication 29x12	1.00	.808	28	1.40	.782	34	2.51*
2. Division 144/8	1.12	.718	24	1.66	.895	50	3.33*
6. Division by 0	1.14	.495	20	1.36	.525	34	2.15*
14. Division of 0 Equal	1.56	1.14	36	1.88	1.10	50	1.42
5. Division Fraction Story	1.08	.829	14	1.42	.835	24	2.04*
10. Division of Fractions	0.88	.773	16	1.26	.723	26	2.53*
19. Fraction Intervals	1.28	1.27	34	2.00	1.17	56	2.92*
8. Ratio Growth	2.00	.202	98	2.00	.452	90	0.00
17. Ratio Equality	1.28	.536	28	2.12	.799	78	6.17*
18. Ratio Pounds	1.36	1.06	40	1.82	.962	52	2.26*
13. Ratio	1.46	.862	38	1.82	.596	72	2.43*
4. Problem Stem	1.68	.891	56	2.50	.735	86	5.02*
Average	1.46	0.80	39.2	1.93	0.76	58.6	

Note. (*) = $p < .05$

The largest gain was made on the addition problem (see Table 4). Teachers were asked to add 65 and 28 using three mathematically different strategies. Only 28% of the teachers were able to think of three methods on the pretest while 100% of them were able to on the posttest.

Large gains were also made on three of the four subtraction problems that dealt with student place value strategies. For instance, initially only 44% of the teachers believed that a student could solve 36 – 19 by subtracting the one's digits first and then the ten's digits: 6 – 9 = -3 and 30 – 10 = 20 so 20 – 3 is 17. Even fewer teachers, 26%, thought this strategy would work for any subtraction problem. Seven months later, all of the teachers believed students could use this strategy to solve this problem and 64% of them justified that it would work for any subtraction problem.

In response to a fraction multiplication word problem, only 38% of the teachers could solve it correctly. Seven months later 72% of them were able to solve it, albeit most using visual representations. In fact, only seven teachers attempted to use the traditional algorithm and three of them failed to attain the correct answer. There were three other problems that dealt with fractions, where teachers made modest gains in their understanding – even after over 10 hours of professional development were spent specifically on these types of problems. Given a division of fraction word problem, 67% of the teachers solved the problem correctly at the beginning and 82% in March. However, only 14% of the teachers could write a story problem for 5 ¼ ÷ ½ and that only improved to 24%. Instead, over half of the responses were a story problem for 5 ¼ x ½. The other problem focused on teachers understanding of an alternative algorithm for dividing fractions. Only 16% of the teachers initially understood that when the numerator and the denominator of a fraction are divisible by the divisor, the problem can be solved by dividing the numerators and then the denominators to produce respectively the resulting fraction. Over ¼ of the teachers understood the strategy in the end.

Overall, there was an average gain of about 20% on the inventory. To understand whether this change was significant, I compared the mean scores from the initial administration in early August of 2004 to the second administration of the inventory in March 2005. As observed in Table 1 the mean score of the teachers' content knowledge is statistically different. A visual difference in the growth of teachers' content knowledge by school over the seven month professional development period is depicted in Figure 1 below through the use of boxplots.

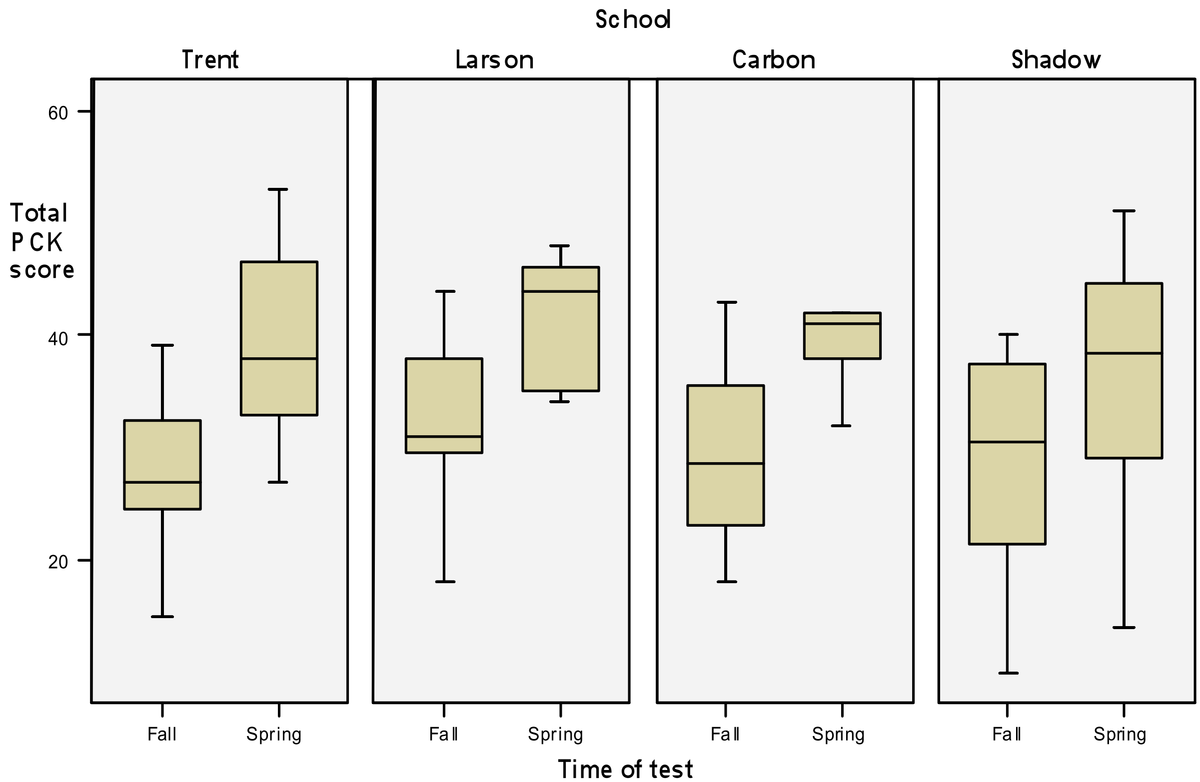
TEACHER CONTENT KNOWLEDGE IN RELATION TO INSTRUCTION

By using the Pearson Correlation I examined a number of relationships among teacher variables, their content knowledge and their instructional practices (see Table 5). Interestingly, I found that neither the number of years a teacher has taught nor their grade level was related to their initial or final content knowledge or to their initial or average instruction practice.

Using classroom observations as an initial starting point, project staff interviewed teachers regarding the number of days a week they devoted to DMT practices. The number of days teachers devoted to DMT practices was not related to their initial content knowledge, $r(50) = .08$, $p = .28$, but was significantly related to teachers' posttest scores $r(50) = .39$, $p < .01$, and significantly related to the gains $r(50) = .35$, $p < .01$, they made from the pretest to the posttest. The number of days teachers taught using DMT methodologies is significantly related $r(50) = .40$, $p < .01$, to teachers' average instructional practices.

This finding is important since the professional development was focused on similar ideas as what the observation scales were detecting. Lacking significance in this category would demonstrate failure in the intent of the professional development.

Figure 1. *Boxplots of Teacher Knowledge Changes from Pretest to Posttest for each School*



When instructional practice is broken down into its six constituent attributes – tasks, role of the teacher, culture of the classroom, tools, equity and accessibility, and discourse – a clearer relationship is observed (see Table 6). First, the number of years a teacher has taught has an inverse relationship to their instructional practices, where the equity promoted in the classroom is statistically significant. Another inverse relationship is between grade level and instruction. In other words, teachers at the lower grade levels were observed to include all students in discussions and promote the sharing of students' ideas (equity and accessibility), to encourage both procedural and conceptual understanding and promote multiple strategies when solving problems (role of the teacher), to value students' ideas and methods, to allow students to choose their own mathematical strategies, to create a safe learning environment, and to encourage students to use mathematical arguments to determine the correctness of the problem (social culture). Teachers in the upper grades were less apt to create learning environments that met these characteristics.

Table 5. *Correlations of Teacher Variables, Content Knowledge and Instruction (n=50)*

	Grade Level	Years Teaching	Days DMT	Initial CK	Final CK	CK Gain
Days DMT	-.134	-.157				
Initial CK	.078	-.094	.084			
Final CK	.201	-.113	.385**	.631**		
CK Gain	.145	-.022	.350**	-.431**	.428**	
Initial Instruction	-.163	-.142	-.090	-.270*	-.068	.249*
Average Instruction	-.076	-.151	.401**	.190	.386**	.229

Note. *p < .05, **p < .01

Table 6. *Correlations Related to Instructional Attributes (n = 49)*

Instruction Attributes	Years Teaching	Days/Week DMT	Grade Level	Content Knowledge	CK Gain
Tasks	-.021	.278*	-.056	.281*	.152
Role of Teacher	-.177	.339*	-.148	.293*	.182
Culture	-.019	.273*	-.051	.234	.107
Tools	-.193	.390**	-.306*	.240*	.219
Equity	-.276*	.419**	-.292*	.372**	.206
Discourse	-.078	.421**	-.023	.349*	.239*

Note. *p < .05, **p < .01.

The number of days DMT was implemented was highly correlated with each aspect of teaching for understanding. The strongest correlations (at a significance level of less than 0.01) were to the types of discourse, equity, and tools promoted in the classroom. The tasks chosen, the role of the teacher, and the culture created were also significantly related to the number of days a teacher implemented the DMT instructional methods.

Finally, there was a significant correlation between content knowledge and the six instructional attributes: tasks was $r(50) = .28$, $p = .025$, role of the teacher was $r(50) = .29$, $p = .020$, classroom culture was $r(50) = .23$, $p = .107$, tools was r(50) = .24, $p = .049$, equity $r(50)$ = =.37, $p = .004$, and discourse was $r(50)$ = 349, $p = .007$. The teachers who had the largest gains encouraged their students discuss mathematical concepts and justify their ideas. These teachers also used more mathematical tools in their classrooms, provided all students the opportunities to learn the mathematics, encouraged reflective dialogue, built a safe culture to make mistakes, and provided students with problems that allowed them to think both procedurally and conceptually.

Discussion, Conclusions and Implications

The central goal of this study was to examine teachers' content knowledge, the change in this knowledge due to a professional development program, and the relationship of content knowledge to instructional practices. It is evident that teachers significantly increased their

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content knowledge after participating in a professional development program focused on developing teachers' and students' mathematical thinking.

Before any professional development began, teachers did not have a deep understanding of number and number operations at either a procedural or conceptual level. Through professional development, teachers were introduced to a continuum of informal and formal strategies to solve problems (Gravemeijer, 2004; Kazemi & Franke, 2003).

The results of this study reveal a moderately strong relationship between teachers' knowledge gains on the number inventory and their instructional practices regarding the teaching of number. This might mean that the teachers who participated in the week-long summer institute on teaching number and, then, attempted to implement these ideas consistently in their classroom increased their content knowledge to a larger degree than teachers who did not attempt to implement these ideas in their classroom.

I also found that Kindergarten, first and second grade teachers were much more willing to try the DMT practices in their classroom. This is evident through the observations of their instruction. These teachers encouraged their students to solve, for instance, addition problems using decomposition (or breaking numbers apart as in 17 + 13 = 10 + 7 and 10 + 3, which allows 10 + 10 + 7 + 3) and compensation (which is changing a number to make the operation easier and then modifying the result at the end of the operation; e.g., 17 + 13 = 20 + 13 – 3) strategies and observed first hand that their students could not only solve the problems correctly, but were also able to conceptually describe the process. By supporting students' construction of knowledge through activities that promoted mathematical analysis, their students made deep connections between different types of mathematics and operations. These findings are consistent with conclusions found in the research literature (Borko et al., 1992; Fennema et al., 1996; Mosenthal, 1995; Onosko, 1990, 1991; Thompson, 1992; Wilson, 1994).

Intermediate grade teachers were mixed and at times resisted teaching for understanding stating that they had to still prepare their students to take standardized tests, and that meant to them teaching the traditional algorithms. Their content knowledge was slightly stronger, but their instructional practices did not focus on the attributes for teaching for understanding. This argues that content knowledge alone may not be the best indicator to focus on for improving teachers' instructional practices.

Another finding is that the more the teachers' knowledge improved the more she or he taught mathematics conceptually. One reason for this is that the professional development model focused on aspects of teaching for understanding, (Cohen, McLaughlin, & Talbert, 1993; Hiebert & Carpenter, 1992) which is what the observation instrument paralleled.

Sherin (2002) described three changes typically seen when teachers try to shift from the traditional teaching of mathematics to an approach that falls more in line with teaching for understanding. The first change is seen in the instructional materials teachers use. However, much of the DMT focused on enabling teachers to create worthwhile tasks to help students explore mathematical relationships. Teachers in our program created these activities themselves with assistance from colleagues.

The second change Sherin (2002) found was in instructional practices. I found that, indeed, teachers' practices changed to focus more on teaching for understanding. The third change was an increased focus on classroom discourse (Brendefur & Frykholm, 2000; Kazemi, 1998; Kazemi & Stipek, 2001). I found that discourse was significantly related to gains teachers made on the content knowledge inventory. When teachers were observed having dialogic conversations, I noted more students were involved in making mathematical connections and defending their ideas mathematically to other students. It follows that when teachers are pressed to conceptualize the mathematics themselves through professional development, their understanding of the mathematics increases and they, then, model similar styles of teaching in their own classrooms.

This study suggests that elementary teachers' content and pedagogical content knowledge can be increased through continuous professional development experiences that focus on developing mathematical thinking. When teachers know how to implement mathematics through a conceptual and procedural manner and one in which students' informal and formal ideas are used to guide instruction, their own knowledge of mathematics increases and their instructional practices improve.

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APPENDIX A

Content Knowledge Inventory – Number

1. What are at least 3 different ways that children might solve 18 x 25?[i]
2. 144 divided by 8. A student says, "I will keep dividing by 2. I need to do this 4 times, because
   2 + 2 + 2 + 2 = 8.[ii]
   144
   / 2
   72
   / 2
   36
   / 2
   18
   / 2
   9 "The answer is 9."

   How would you respond to this student?
3. A student says she has figured out an easy way to solve double-digit subtraction problems.[iii]

36	6 – 9 = -3	
- 19	30 – 10 = 20	
	-3 + 20 = 17	"The answer is 17."

   Does this strategy make sense? Why or why not?
   If this strategy works, would it work for all subtraction problems all the time? Why or why not?
4. Problem: There is a bag on the table with 3 blue marbles and 8 red marbles.
   What question(s) might you ask students based on this problem stem? Why?
5. Write a story problem for the following division problem: 5 ¼ ÷ ½.[iv]
6. Suppose you have a student who asks you what 7 divided by 0 is. How would you respond?[v]
7. Some sixth grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate 123 x 645. The students seemed to be forgetting to "move the numbers" over on each line. They were doing this: Instead of this:

They were doing this:	Instead of this:
123	123
x 645	x 645
615	615
492	492
738	738___

1845 79335

While these teachers agreed that this was a problem, they did not agree on what to do about it. What would you do if you were teaching sixth grade and you noticed that several of your students were doing this?[vi]

8. Jo has 2 snakes, String Bean and Slim. Right now, String Bean is 4 feet long and Slim is 5 feet long. Jo knows that 2 years from now both snakes will be fully grown. At her full length, String Bean will be 7 feet long, while Slim's length when he is fully grown will be 8 feet long. Over the next 2 years, will both snakes grow the same amount? Explain.[vii]
9. Look at these two problems:

$$\begin{array}{r} 52 \\ \underline{-\ 25} \end{array} \qquad \begin{array}{r} 91 \\ \underline{-\ 79} \end{array}$$

   How would you approach these problems if you were teaching second grade? What would you say pupils would need to understand or be able to do before they could start learning subtraction with regrouping?

10. A student says she has figured out an easy way to divide fractions.[viii]

$$1\tfrac{3}{4} \div \tfrac{1}{2} = \frac{7}{4} \div \frac{1}{2} = \frac{7 \div 1}{4 \div 2} = \frac{7}{2} = 3\tfrac{1}{2}$$

   Does this strategy make sense? Why or why not?

   If this strategy works, would it work for all division by fraction problems? Why or why not?

11. One of your students is having trouble subtracting fractions with regrouping. He is continually solving problems like this:

$$\begin{array}{rcl} 3\ 2/5 & = & 2\ 12/5 \\ \underline{-\ 2\ 4/5} & = & \underline{2\ 4/5} \\ & & 8/5 = \boxed{1\ 3/5} \end{array}$$

   What misconception does this student have? How would you help this student?

12. Wanda really likes cake. She has decided that a serving should be 3/5 of a cake. If she orders four cakes, how many servings can she make?[ix]
13. Are the following fractions equivalent?[x] 0/3 and 0/7
14. One of your students shows the class a new method for multiplying 29 × 12: [xi]

   29 + 1 = 30

(*because 30 is easier than 29*)

$$\begin{array}{r} 30 \\ \underline{\times\ 12} \\ 360 \\ \underline{-\ 12} \\ 348 \end{array}$$

(*because I added one 12 in the beginning*)

So another student tries to use the same method on the next problem, 36 × 17:

   36 + 4 = 40

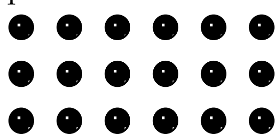
   17 + 3 = 20

$$\begin{array}{r} 40 \\ \underline{\times\ 20} \end{array}$$

```
 800
 - 4   (because I added 4 to make 40)
 796
 - 3   (because I added 3 to make 20)
 793
```

Is there any sense in this student's method? What mathematics is missing in this student's method?

15. Use the picture below to answer the questions:[xii]



a) Can you see thirds? How many dots in 2/3 of the set?
b) Can you see sixths? How many dots in 5/6 of the set?
c) Can you see ninths? How many dots in 7/9 of the set?
d) Can you see twelfths? How many dots in 7/12 of the set?
e) Can you see eighteenths? How many dots in 11/18 of the set?

16. What are at least 3 different ways that children might solve 65 + 28?
17. What are the different responses that students may give to the following open number sentence:

    $9 + 7 = \square + 8$? [xiii]

18. Marissa bought 0.43 pounds of wheat flour for which she paid $0.86. How many pounds of flour could she buy for $1.00?
19. What fraction is between 1/4 and 1/5?

Appendix B

DMT Observation Scoring Sheet

Teacher: **Grade Level:** **Date:**

Attribute	Description	Scoring
		None Little Most All **0------1-------2-------3**
Nature of Classroom Tasks	Connects with where students are	
	Leaves behind something of mathematical value	
	Requires students to use HOT when addressing the concept or problem	
	Provides opportunities for students to consider alternative strategies	
Role of the Teacher	Select tasks with goals in mind	
	Share essential information	
	Uses combination of conceptual and procedural knowledge	
	Provides opportunities for students to consider alternative strategies	
Social Culture of the Classroom	Ideas and methods are valued	
	Students choose and share their methods	
	Mistakes and misconceptions are learning opportunities for everyone	
	Correctness resides in mathematical argument	
	Safe learning environment	
Mathematical Tools as Learning Supports	Meaning for tools must be constructed by each user	
	Used with purpose—to solve problems	
	Used for recording, communicating, and thinking	
Equity and Accessibility	Tasks are accessible to all students	
	Every student is heard	
	All contribute	
Classroom Discourse	Elaborate explanations	
	Justify solution strategies – (logical argument)	
	Conceptual Press – Univocal and dialogical	

[i] Empson & Junk (2004) p.126
[ii] Empson & Junk (2004) p.126
[iii] Empson & Junk (2004) p.127
[iv] Kennedy, Ball, & McDiarmid (1993) p.49; Ball (1989) p.75; Ball & Wilson (1990) p.4; Ma (1999) p.55
[v] Kennedy, Ball, & McDiarmid (1993) p.53; Ball & Wilson (1990) p.4; Ma (1999) p.28
[vi] Ma (1999) p. 28; Kennedy, Ball, & McDiarmid (1993) p.38; Ball, Lubienski, & Mewborn (2001) p.444
[vii] Lamon (1999) p.12; Sowder, Philipp, Armstrong, & Schappelle (1998)
[viii] Ma (1999) p.63
[ix] Schifter(1998) p.68; Schifter(2001) p.112
[x] Lamon (1999) p.73
[xi] Schifter(2001) p.118

[xii] Lamon (1999) p.73
[xiii] Carpenter, Franke, & Levi (2003) p.24